DURHAM UNIVERSITY

Mathematical Sciences Department

MSc Particles, Strings and Cosmology

General Relativity

Homework

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Problem 1. Show that if k^μ is a Killing vector and $M_{\mu\nu} = \nabla_{[\mu}k_{\nu]}$ then $\nabla_\mu M^{\mu\nu} = 0$ provided the background geometry satisfies Einstein's equation in the vacuum.

Solution 1. Since k^ν is a killing vector, then it satisfy

$$\nabla_{(\mu}k_{\nu)} = \frac{1}{2}(\nabla_\mu k_\nu + \nabla_\nu k_\mu) = 0, \qquad \Longleftrightarrow \qquad \nabla_\mu k_\nu = -\nabla_\nu k_\mu.$$

While

$$M_{\mu\nu} = \nabla_{[\mu}k_{\nu]} = \frac{1}{2}(\nabla_\mu k_\nu - \nabla_\nu k_\mu) = \nabla_\mu k_\nu,$$

where the last term is the result of plugging the fact that k^ν is a killing vector, i.e. $-\nabla_\nu k_\mu = \nabla_\mu k_\nu$. Now, the condition $\nabla_\mu M^{\mu\nu} = 0$ implies that

$$\nabla_\mu \nabla^\mu k^\nu = 0,$$

and this is our first result. The antisymmetricity in the last two indices says

$$\nabla_\mu \nabla^\mu k^\nu = \frac{1}{2}\nabla_\mu(\nabla^\mu k^\nu - \nabla^\nu k^\mu).$$

Now, by definition with tortion-free,

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho, \qquad \text{or,} \qquad \nabla_\mu\nabla_\nu V^\lambda = \nabla_\nu\nabla_\mu V^\lambda + R^\lambda{}_{\rho\mu\nu}V^\rho,$$

Applying this relation to the equation we work at,

$$\nabla_\mu \nabla^\mu k^\nu = \frac{1}{2}(\nabla_\mu \nabla^\mu k^\nu - \nabla^\nu \nabla_\mu k^\mu - R^\mu{}_{\rho\mu}{}^\nu k^\rho).$$

The right hand side is zero from our first result, the first term in the left hand side is also zero by the same reason, the second term of the left hand side will also equal to zero by Killing's equation. Hence, the last term will be zero since all other terms are zeros, and with using the contraction of the Riemann tensor to have the Ricci tensor, i.e. $R^\mu{}_{\rho\mu}{}^\nu = R_\rho{}^\nu$, then

$$R_\rho{}^\nu k^\rho = 0.$$

This equation tells us that our Ricci tensor is for flat spacetime, and hence it satisfies Einstein's equation in the absence of matter fields. In other words, the background geometry satisfies Einstein's equation in the vacuum. Mathematically speaking, for general Killing vector k^ρ, then the final result implies that

$$R_\rho{}^\nu = 0,$$

and consequently $R = R_\rho{}^\rho = 0$. Hence, for Einstein's field equation, which is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu},$$

then by substitution of the Ricci tensor and the curvature, we get

$$T_{\mu\nu} = 0,$$

and the Einstein's field equation describes the background geometry in the vacuum, i.e.

$$G_{\mu\nu} = 0,$$

and remember that all this because $M_{\mu\nu} = \nabla_{[\mu} k_{\nu]}$. ■

Problem 2. Consider the covariant actions of a scalar field φ and Maxwell 1-form field A_μ given by

$$\begin{aligned} S_{\text{scalar}} &= \int \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V(\varphi) \right) \\ S_{\text{Maxwell}} &= \int \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \end{aligned}$$

with V any real function and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Derive the equations of motion for φ and A_μ based on these actions.

(b) Derive an expression for the corresponding stress tensors $T^{\text{scalar}}_{\mu\nu}$ and $T^{\text{Maxwell}}_{\mu\nu}$ by considering the variation of these actions with respect to the background metric.

(c) Check explicitly that $\nabla_\mu T^{\mu\nu} = 0$ in both cases by using the equations you derived in the first question.

Solution 2.

(a) The Euler-Lagrange equation for the scalar field φ is given by

$$\frac{\partial \mathcal{L}^0_{\text{scalar}}}{\partial \varphi} - \nabla_\mu \left(\frac{\partial \mathcal{L}^0_{\text{scalar}}}{\partial (\nabla_\mu \varphi)} \right) = 0, \qquad \text{where} \qquad \mathcal{L}^0_{\text{scalar}} = -\frac{1}{2}\nabla_\mu \varphi \nabla^\mu \varphi - V(\varphi).$$

then the equation of motion for the scalar field φ is given by

$$\nabla_\mu \nabla^\mu \varphi - \frac{dV(\varphi)}{d\varphi} = 0.$$

The same process for the field A_μ,

$$\frac{\partial \mathcal{L}^0_{\text{Maxwell}}}{\partial A_\sigma} - \nabla_\rho \left(\frac{\partial \mathcal{L}^0_{\text{Maxwell}}}{\partial (\nabla_\rho A_\sigma)} \right) = 0, \qquad \text{where} \qquad \mathcal{L}^0_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

The first term of Euler-Lagrange equation for the field A_μ is automatically zero, because the Lagrangian $\mathcal{L}^0_{\text{Maxwell}}$ does not depend explicitly on it. Now, the second part, we have a derivative with respect to the covariant derivative of Faraday tensor. In going to curved spacetime, we would like to replace partial derivatives according to the Equivalence Principle. This will not affect the value of the electromagnetic field tensor, i.e.

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

Hence,

$$\frac{\partial \mathcal{L}^0_{\text{Maxwell}}}{\partial (\nabla_\rho A_\sigma)} = -F^{\rho\sigma},$$

and therefore, the equation of motion for the field A_μ is given by

$$\nabla_\rho F^{\rho\sigma} = 0.$$

(b) The stress tensor for the scalar action with respect to the background metric is given by

$$T^{\text{scalar}}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{scalar}}}{\delta g^{\mu\nu}},$$

where

$$\frac{\delta S_{\text{scalar}}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \mathcal{L}^0_{\text{scalar}} + \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \varphi \nabla_\nu \varphi \right),$$

and therefore,

$$T^{\text{scalar}}_{\mu\nu} = g_{\mu\nu} \mathcal{L}^0_{\text{scalar}} + \nabla_\mu \varphi \nabla_\nu \varphi.$$

On the other hand, the stress tensor for the Maxwell action with respect to the background metric is given by

$$T^{\text{Maxwell}}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{Maxwell}}}{\delta g^{\mu\nu}},$$

where
$$\frac{\delta S_{\text{Maxwell}}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\mathcal{L}^0_{\text{Maxwell}} - \frac{\sqrt{-g}}{4}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} + F_{\lambda\mu}F^{\lambda}{}_{\nu}\right),$$
and therefore, (not forgetting the antisymmetricity of Faraday tensor)
$$T^{\text{Maxwell}}_{\mu\nu} = g_{\mu\nu}\mathcal{L}^0_{\text{Maxwell}} + F_{\mu\lambda}F_{\nu}{}^{\lambda}.$$

(c) Explicitly checking:

$$\begin{aligned}
\nabla_\mu T^{\mu\nu}_{\text{scalar}} =& \nabla_\mu\left(g^{\mu\nu}\mathcal{L}^0_{\text{scalar}} + \nabla^\mu\varphi\nabla^\nu\varphi\right)\\
=&\nabla^\nu\left(-\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi)\right) + \nabla_\mu\left(\nabla^\mu\varphi\nabla^\nu\varphi\right)\\
&\qquad\qquad [\nabla_\mu g^{\mu\nu} = 0 \text{ and } g^{\mu\nu}\nabla_\mu = \nabla^\nu]\\
=& -\nabla^\mu\varphi\nabla^\nu\nabla_\mu\varphi - \nabla^\nu V(\varphi) + \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi + \nabla^\mu\varphi\nabla^\nu\nabla_\mu\varphi\\
=&\left(\nabla_\mu\nabla^\mu\varphi - \frac{dV(\varphi)}{d\varphi}\right)\nabla^\nu\varphi \qquad \left[\nabla^\nu V(\varphi) = \frac{dV(\varphi)}{d\varphi}\nabla^\nu\varphi\right]\\
=&0 \qquad [\text{EOM}]
\end{aligned}$$

$$\begin{aligned}
\nabla_\mu T^{\mu\nu}_{\text{Maxwell}} =& \nabla_\mu\left(g^{\mu\nu}\mathcal{L}^0_{\text{Maxwell}} + F^\mu{}_\lambda F^{\nu\lambda}\right)\\
=&\nabla^\nu\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) + \nabla_\mu\left(F^\mu{}_\lambda F^{\nu\lambda}\right)\\
=& -\frac{1}{2}F_{\alpha\beta}\nabla^\nu F^{\alpha\beta} + \nabla_\mu F^\mu{}_\lambda F^{\nu\lambda} + F^\mu{}_\lambda\nabla_\mu F^{\nu\lambda}\\
=& -\frac{1}{2}F_{\alpha\beta}\nabla^\nu F^{\alpha\beta} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} \qquad [\text{EOM}]\\
=&F_{\alpha\beta}\nabla^\alpha F^{\nu\beta} + \frac{1}{2}F_{\alpha\beta}\nabla^\nu F^{\beta\alpha} \qquad [\text{antisym. of } F^{\alpha\beta}]\\
=&\frac{1}{2}F_{\alpha\beta}\left(\nabla^\alpha F^{\nu\beta} - \nabla^\beta F^{\nu\alpha} + \nabla^\nu F^{\beta\alpha}\right)\\
=&0 \qquad [\text{Bianchi identity}]
\end{aligned}$$