DURHAM UNIVERSITY
Mathematical Sciences Department
MSc Particles, Strings and Cosmology
Group Theory
Homework

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Problem 1. The tetrahedral group T is the group of orientation-preserving symmetries of a regular (equilateral) tetrahedron. If the vertices of the tetrahedron are labeled A,B,C,D, each of the orientation-preserving symmetries may be represented as an even permutation of these four symbols. There are 12 of them.

- The identity I.
- Eight rotations about axes going through a vertex and the opposite face. There are four vertices; each rotation can be by 120 degrees or 240 degrees:
  - (BCD) and (BDC) are rotations of 120 degrees and 240 degrees resp. about an axis through A and the centre of the face opposite to A
  - (ADC) and (ACD) are rotations of 120 degrees and 240 degrees resp. about an axis through B and the centre of the face opposite to B
  - (ADB) and (ABD) are rotations of 120 degrees and 240 degrees resp. about an axis through C and the centre of the face opposite to C
  - (ACB) and (ABC) are rotations of 120 degrees and 240 degrees resp, about an axis through D and the centre of the face opposite to D

  We shall label the 120 degree rotations as $R_1, \dots, R_4$, and the 240 degree rotations as $R_1^2, \dots, R_4^2$. Note that all these permutations of vertices are even, i.e. they can be written as the composition of an even number of transpositions. E.g. $(ADC) = (AD)(DC)$.
- Three rotations of 180 degrees about axes joining the midpoints of two opposite edges: $(AB)(CD)$, $(AC)(BD)$ and $(AD)(BC)$. We shall label these as C_{2x}, C_{2y} and C_{2z}.

Given the multiplication table for T below,

$\circ$	I	C_{2x}	C_{2y}	C_{2z}	R_1	R_2	R_3	R_4	R_1^2	R_3^2	R_4^2	R_2^2
I	I	C_{2x}	C_{2y}	C_{2z}	R_1	R_2	R_3	R_4	R_1^2	R_3^2	R_4^2	R_2^2
C_{2x}	C_{2x}	I	C_{2z}	C_{2y}	R_2	R_1	R_4	R_3	R_3^2	R_1^2	R_2^2	R_4^2
C_{2y}	C_{2y}	C_{2z}	I	C_{2x}	R_3	R_4	R_1	R_2	R_4^2	R_2^2	R_1^2	R_3^2
C_{2z}	C_{2z}	C_{2y}	C_{2x}	I	R_4	R_3	R_2	R_1	R_2^2	R_4^2	R_3^2	R_1^2
R_1	R_1	R_3	R_4	R_2	R_1^2	R_4^2	R_2^2	R_3^2	I	C_{2y}	C_{2z}	C_{2x}
R_3	R_3	R_1	R_2	R_4	R_4^2	R_1^2	R_3^2	R_2^2	C_{2y}	I	C_{2x}	C_{2z}
R_4	R_4	R_2	R_1	R_3	R_2^2	R_3^2	R_1^2	R_4^2	C_{2z}	C_{2x}	I	C_{2y}
R_2	R_2	R_4	R_3	R_1	R_3^2	R_2^2	R_4^2	R_1^2	C_{2x}	C_{2z}	C_{2y}	I
R_1^2	R_1^2	R_2^2	R_3^2	R_4^2	I	C_{2z}	C_{2x}	C_{2y}	R_1	R_4	R_2	R_3
R_2^2	R_2^2	R_1^2	R_4^2	R_3^2	C_{2z}	I	C_{2y}	C_{2x}	R_4	R_1	R_3	R_2
R_3^2	R_3^2	R_4^2	R_1^2	R_2^2	C_{2x}	C_{2y}	I	C_{2z}	R_2	R_3	R_1	R_4
R_4^2	R_4^2	R_3^2	R_2^2	R_1^2	C_{2y}	C_{2x}	C_{2z}	I	R_3	R_2	R_4	R_1

(a) What are the conjugacy classes of T?

(b) What are the subgroups of T?

(c) What are the invariant subgroups of T and their right cosets?

(d) Consider the unique non trivial, proper invariant subgroup of T and call it K (it should be on your list in part (c)). What is the index of K in T? Construct the Cayley table for the quotient group $(T/K, \circ)$. Do you recognise a familiar group?

(e) Construct a one-to-one mapping

$$\vartheta : T/K \to M_2(\mathbb{R})$$

of the right cosets of K onto a set of 2×2 matrices with real entries. What is $\ker\vartheta$?

(f) Using the multiplication table of T, explain why the non-invariant subgroup $S = \{I, C_{2x}\}$ of T cannot be used as the kernel of a mapping $\varphi : T/S \to M_2(\mathbb{R})$.

Solution 1.

(a) The conjugacy classes of T can be founded easily from Cayley table, which are the following 4 classes:

$$Cl(T) = \{\{I\}, \{C_{2x}, C_{2y}, C_{2z}\}, \{R_1, R_2, R_3, R_4\}, \{R_1^2, R_2^2, R_3^2, R_4^2\}\}.$$

(b) Let's divide the subgroups of the tetrahedral group into classes according to their cardinalities:

- *n = 1: we have 1 subgroup, the trivial group* $\{I\}$.
- *n = 2: we have 3 subgroups:* $\{I, C_{2x}\}, \{I, C_{2y}\}, \{I, C_{2z}\}$.
- *n = 3: we have 4 subgroups:* $\{I, R_1, R_1^2\}, \{I, R_2, R_2^2\}, \{I, R_3, R_3^2\}, \{I, R_4, R_4^2\}$.
- *n = 4: we have 1 subgroup:* $\{I, C_{2x}, C_{2y}, C_{2z}\}$
- *n = 6: we don't have any subgroup.*
- *n = 12: we have 1 subgroup, the trivial group* T.

(c) *The normal subgroups and their right cosets are:*

- *n = 1: we have 1 normal subgroup, and the trivial group* $\{I\}$*, the right cosets are:*

$$\{I\}t = \{t\}, \forall t \in T.$$

- *n = 4: we have 1 normal subgroup:* $\{I, C_{2x}, C_{2y}, C_{2z}\}$*, and the right cosets are:*

$$\{I, C_{2x}, C_{2y}, C_{2z}\}, \{R_1, R_2, R_3, R_4\}, \{R_1^2, R_2^2, R_3^2, R_4^2\}$$

- *n = 12: we have 1 normal subgroup, the trivial group* T*, and the cosets are itself.*

The other cardinalities of subgroups do not have any invariant subgroup.

(d) *The subgroup* $K = \{I, C_{2x}, C_{2y}, C_{2z}\}$*, and therefore its index in* T *is* $|T|/|K| = 12/4 = 3$*. The quotient group* $(T/K, \circ)$ *is*

$$\begin{aligned} T/K &= \{\{I, C_{2x}, C_{2y}, C_{2z}\}, \{R_1, R_2, R_3, R_4\}, \{R_1^2, R_2^2, R_3^2, R_4^2\}\} \\ &= \{E, R, R^2\}, \end{aligned}$$

and hence, the Cayley table of the quotient group $(T/K, \circ)$ *is*

$\circ$	E	R	R^2
E	E	R	R^2
R	R	R^2	E
R^2	R^2	E	R

So, $(T/K, \circ) \simeq (Z_3, \oplus_3)$*; the cyclic group* $\{0, 1, 2\}$ *with addition modulo* 3 *operation* $(\oplus_3)$[1].

[1] This is the notation of the addition modulo an integer number I took in my Abstract Algebra course at my Undergraduate studies, i.e. the operation $(\oplus_n)$ is the addition modulo n.

(e) *One can write the one to one mapping ϑ as*

$$\begin{aligned}\vartheta : (T/K, \circ) &\to (M_2(\mathbb{R}), \cdot)\\ E &\to \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\ R &\to \mathbb{R}(2\pi/3) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\\ R^2 &\to \mathbb{R}(4\pi/3) = \begin{pmatrix} \cos(4\pi/3) & -\sin(4\pi/3) \\ \sin(4\pi/3) & \cos(4\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\end{aligned}$$

$\ker\vartheta = \{E\}$.

(f) *If $S = \ker\varphi$, then it must be invariant under T/S. Note that T/S is not a group, but it still form some algebraic structure, so we just need to deal with the problem in the point of view of algebraic structures in general. However, the definition of T/S itself is unclear in this case, because the left and right cosets are not equal. Anyway, lets start with considering that T/S form the left cosets of S. An equivalent definition of the invariance, if $A \in T/S$ then $A \circ \ker\varphi = \ker\varphi \circ A$, $\forall A \in T/S$. Now, choose $A := R_3 \circ S = \{R_1, R_3\}$, then, $A \circ S = A$, but $S \circ A = A \cup \{R_1, R_4\} \neq A \circ S$. So $S \neq \ker\varphi$ by contradiction with the definition of the kernel. In terms of considering T/S being the right cosets, just use the same technique of the example in the left cosets, e.g. let $B := S \circ R_3$, then, $S \circ B = B$, but $B \circ S = B \cup \{R_1, R_4\} \neq S \circ B$. So $S \neq \ker\varphi$ again by contradiction with the definition of the kernel. Finally, if T/S is the set of left and right cosets of the non-invariant subgroup S, then one of the previous two cases is enough to prove that $S \neq \ker\varphi$.*

Problem 2. Let (x_1, y_1) and (x_2, y_2) be the coordinates of two vectors which transform independently under D_3 transformations as in Exercise 2.6 of the exercise sheet, i.e.

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \mathfrak{m}^{(1)}(g) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \mathfrak{m}^{(2)}(g) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

with $\mathfrak{m}^{(1)}(g) = \mathfrak{m}^{(2)}(g)$ a 2-dimensional representation of D_3 where

$$\mathfrak{m}^{(1)}(r) = \mathfrak{m}^{(2)}(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \qquad \mathfrak{m}^{(1)}(d) = \mathfrak{m}^{(2)}(d) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the function space V_f spanned by the monomials $\{x_1x_2, x_1y_2, x_2y_1, y_1y_2\}$. Show that the representation of D_3 on this 4-dimensional space is the direct product representation $\mathfrak{m}^{(1)} \otimes \mathfrak{m}^{(2)}$.

Solution 2. *Lets treat the representation matrices $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ as different to prove the direct product relation for the representation of D_3 on the 4-dimensional space, i.e. $\mathfrak{m}^{(1)}\otimes\mathfrak{m}^{(2)}$. Firstly, write a function $f : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ as the following,*

$$f(v_1, v_2) = ax_1x_2 + bx_1y_2 + cy_1x_2 + dy_1y_2.$$

This expression can also be written as

$$f(v_1, v_2) = v_1^T A v_2,$$

where

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

The 2-dimensional representation of D_3 acts on the carrier space as

$$\begin{aligned} \mathfrak{m}^{(1)}(g) &: V_2 \to V_2 \qquad & \mathfrak{m}^{(2)}(g) &: V_2 \to V_2 \\ v_1 &\mapsto v_1' = \mathfrak{m}^{(1)}(g)v_1 & v_2 &\mapsto v_2' = \mathfrak{m}^{(2)}(g)v_2 \end{aligned}$$

Hence,

$$f(v_1, v_2) \to (v_1')^T A v_2' = v_1^T \underbrace{(\mathfrak{m}^{(1)}(g))^T A \mathfrak{m}^{(2)}(g)}_{:=A'(g)} v_2,$$

where

$$A'(g) := \begin{pmatrix} a'(g) & b'(g) \\ c'(g) & d'(g) \end{pmatrix}.$$

Let us calculate $a'(g), \dots, d'(g)$ using the general matrices

$$\mathfrak{m}^{(1)}(g) = \begin{pmatrix} m_{11}^{(1)}(g) & m_{12}^{(1)}(g) \\ m_{21}^{(1)}(g) & m_{22}^{(1)}(g) \end{pmatrix}, \qquad \mathfrak{m}^{(2)}(g) = \begin{pmatrix} m_{11}^{(2)}(g) & m_{12}^{(2)}(g) \\ m_{21}^{(2)}(g) & m_{22}^{(2)}(g) \end{pmatrix}$$

in $A'(g) = (\mathfrak{m}^{(1)}(g))^T A \mathfrak{m}^{(2)}(g)$ and collect the resulting information in a 4-dimensional matrix $\mathfrak{M}(g)$. With $w' = (a'(g), \dots, d'(g))^T$, we have

$$w' = \mathfrak{M}(g)w,$$

we will get $\mathfrak{M} = {\mathfrak{m}^{(1)}}^T \otimes \mathfrak{m}^{(2)}$. However, the representation ${\mathfrak{m}^{(1)}}^T \simeq \mathfrak{m}^{(1)}$, and therefore, one can write ${\mathfrak{m}^{(1)}}^T \otimes \mathfrak{m}^{(2)} = \mathfrak{M} \simeq \mathfrak{M}' = \mathfrak{m}^{(1)} \otimes \mathfrak{m}^{(2)}$, with vectors $w'' = (a''(g), \dots, d''(g))^T$ s.t.

$$w'' = \mathfrak{M}'w,$$

e.g. for $a''(g)$,

$$a''(g) = m_{11}^{(1)}(g)m_{11}^{(2)}(g)a + m_{11}^{(1)}(g)m_{21}^{(2)}(g)b + m_{21}^{(1)}(g)m_{11}^{(2)}(g)c + m_{21}^{(1)}(g)m_{21}^{(2)}(g),$$

and completing for the other coefficients, we get

$$
\begin{aligned}
\mathfrak{M}'(g) = &\begin{pmatrix}
m^{(1)}_{11}(g)m^{(2)}_{11}(g) & m^{(1)}_{11}(g)m^{(2)}_{12}(g) & m^{(1)}_{12}(g)m^{(2)}_{11}(g) & m^{(1)}_{12}(g)m^{(2)}_{12}(g) \\
m^{(1)}_{11}(g)m^{(2)}_{21}(g) & m^{(1)}_{11}(g)m^{(2)}_{22}(g) & m^{(1)}_{12}(g)m^{(2)}_{21}(g) & m^{(1)}_{12}(g)m^{(2)}_{22}(g) \\
m^{(1)}_{21}(g)m^{(2)}_{11}(g) & m^{(1)}_{21}(g)m^{(2)}_{12}(g) & m^{(1)}_{22}(g)m^{(2)}_{11}(g) & m^{(1)}_{22}(g)m^{(2)}_{12}(g) \\
m^{(1)}_{21}(g)m^{(2)}_{21}(g) & m^{(1)}_{21}(g)m^{(2)}_{22}(g) & m^{(1)}_{22}(g)m^{(2)}_{21}(g) & m^{(1)}_{22}(g)m^{(2)}_{22}(g)
\end{pmatrix} \\
=&\mathfrak{m}^{(1)} \otimes \mathfrak{m}^{(2)}.
\end{aligned}
$$

Now, explicitly, if $\mathfrak{m}^{(1)} = \mathfrak{m}^{(2)} =: \mathfrak{m}$, *using the 2-dimensional representations given in the problem and the result obtained in the proof, then*

$$
\mathfrak{M}'(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad
\mathfrak{M}'(r) = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ \sqrt{3} & 1 & -3 & -\sqrt{3} \\ \sqrt{3} & -3 & 1 & -\sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix},
$$

$$
\mathfrak{M}'(r^2) = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}, \qquad
\mathfrak{M}'(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\mathfrak{M}'(rd) = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ -\sqrt{3} & -1 & 3 & \sqrt{3} \\ -\sqrt{3} & 3 & -1 & \sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix}, \qquad
\mathfrak{M}'(r^2d) = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ \sqrt{3} & -1 & 3 & -\sqrt{3} \\ \sqrt{3} & 3 & -1 & -\sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}.
$$

Problem 3. Let $a, b, c, d \in \{1, \ldots, n\}$. Consider the n^2 matrices ${M_a}^b$ with matrix elements given by

$$({M_a}^b)_{cd} := \delta_{ac}\delta_d{}^b - \frac{1}{n}\delta_a{}^b\delta_{cd}$$

(i) Calculate the trace of these matrices.

(ii) What are their commutation relations, i.e. what is their Lie bracket $[{M_a}^b, {M_c}^d]$?

(iii) Give the matrix form of all the ${M_a}^b$'s in the case where $n = 2$ and find which (complex) simple Lie algebra they generate. Do they form a basis of that Lie algebra?

(iv) Show that in the case $n = 2$, the differential operators $2jx - x^2\frac{d}{dx}$, $\frac{d}{dx}$ and $x\frac{d}{dx} - j$ where $j \in \mathbb{N}$ and $x \in \mathbb{R}$ provide another representation of the Lie algebra found in part (iii).

Solution 3.

(i) The trace of the matrices ${M_a}^b$ is

$$\begin{aligned}\mathrm{Tr}({M_a}^b) = \sum_{k=1}^{n} ({M_a}^b)_{kk} = \sum_{k=1}^{n} \left(\delta_{ak}{\delta_k}^b - \frac{1}{n}{\delta_a}^b \delta_{kk}\right) = {\delta_a}^b - \sum_{k=1}^{n} \frac{1}{n}{\delta_a}^b \\ ={\delta_a}^b - {\delta_a}^b = 0.\end{aligned}$$

(ii) Lets first find the matrix multiplication ${M_a}^b {M_c}^d$,

$$\begin{aligned}({M_a}^b {M_c}^d)_{ij} &= \sum_{k=1}^{n} ({M_a}^b)_{ik} ({M_c}^d)_{kj} \\ &= \sum_{k=1}^{n} \left(\delta_{ai}{\delta_k}^b - \frac{1}{n}{\delta_a}^b \delta_{ik}\right) \left(\delta_{ck}{\delta_j}^d - \frac{1}{n}{\delta_c}^d \delta_{kj}\right) \\ &= \sum_{k=1}^{n} \left(\delta_{ai}{\delta_k}^b \delta_{ck}{\delta_j}^d - \frac{1}{n}({\delta_a}^b \delta_{ik}\delta_{ck}{\delta_j}^d + \delta_{ai}{\delta_k}^b {\delta_c}^d \delta_{kj}) + \frac{1}{n^2}{\delta_a}^b \delta_{ik}{\delta_c}^d \delta_{kj}\right) \\ &= \delta_{ai}{\delta_c}^b {\delta_j}^d - \frac{1}{n}({\delta_a}^b \delta_{ic}{\delta_j}^d + \delta_{ai}{\delta_j}^b {\delta_c}^d) + \frac{1}{n^2}{\delta_a}^b \delta_{ij}{\delta_c}^d\end{aligned}$$

with the same analogy,

$$({M_c}^d {M_a}^b)_{ij} = \delta_{ci}{\delta_a}^d {\delta_j}^b - \frac{1}{n}({\delta_c}^d \delta_{ia}{\delta_j}^b + \delta_{ci}{\delta_j}^d {\delta_a}^b) + \frac{1}{n^2}{\delta_c}^d \delta_{ij}{\delta_a}^b.$$

Then the commutation relation $[{M_a}^b, {M_c}^d]_{ij}$ is

$$\begin{aligned}[{M_a}^b, {M_c}^d]_{ij} &= ({M_a}^b {M_c}^d)_{ij} - ({M_c}^d {M_a}^b)_{ij} \\ &= \delta_{ai}{\delta_c}^b {\delta_j}^d - \delta_{ci}{\delta_a}^d {\delta_j}^b \\ &= \delta_{ai}{\delta_j}^d {\delta_c}^b + \frac{1}{n}\left({\delta_a}^d \delta_{ij} - {\delta_a}^d \delta_{ij}\right){\delta_c}^b - \delta_{ci}{\delta_j}^b {\delta_a}^d + \frac{1}{n}\left({\delta_c}^b \delta_{ij} - {\delta_c}^b \delta_{ij}\right){\delta_a}^d \\ &= \left(\delta_{ai}{\delta_j}^d - \frac{1}{n}{\delta_a}^d \delta_{ij}\right){\delta_c}^b - \left(\delta_{ci}{\delta_j}^b - \frac{1}{n}{\delta_c}^b \delta_{ij}\right){\delta_a}^d + \frac{1}{n}\left({\delta_a}^d {\delta_c}^b - {\delta_a}^d {\delta_c}^b\right)\delta_{ij} \\ &= ({M_a}^d)_{ij}{\delta_c}^b - ({M_c}^b)_{ij}{\delta_a}^d \\ &= ({M_a}^d {\delta_c}^b - {M_c}^b {\delta_a}^d)_{ij},\end{aligned}$$

and hence,

$$[{M_a}^b, {M_c}^d] = {M_a}^d {\delta_c}^b - {M_c}^b {\delta_a}^d.$$

(iii) The matrices ${M_a}^b$ for $a, b \in \{1, 2\}$ are

$${M_1}^1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad {M_1}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$${M_2}^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad {M_2}^2 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

and they spanned the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Obviously, they do not form a basis because ${M_1}^1$ and ${M_2}^2$ are linearly dependent. However, if we introduce the diagonal matrix M_d i.e.

$$M_d = {M_1}^1 - {M_2}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then the set $\mathcal{B} = \{M_d, {M_1}^2, {M_2}^1\}$ will form a basis for the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

(iv) Using the commutation brackets,

$$\begin{aligned}
\left[\left(2jx - x^2\frac{d}{dx}\right), \frac{d}{dx}\right] f &= \left(\left(2jx - x^2\frac{d}{dx}\right)\frac{d}{dx} - \frac{d}{dx}\left(2jx - x^2\frac{d}{dx}\right)\right) f \\
&= \left(\left(2jx\frac{d}{dx} - x^2\frac{d^2}{dx^2}\right)\right. \\
&\quad \left. - \left(2j + 2jx\frac{d}{dx} - 2x\frac{d}{dx} - x^2\frac{d^2}{dx^2}\right)\right) f \\
&= 2\left(x\frac{d}{dx} - j\right) f,
\end{aligned}$$

$$\begin{aligned}
\left[\frac{d}{dx}, \left(x\frac{d}{dx} - j\right)\right] f &= \left(\frac{d}{dx}\left(x\frac{d}{dx} - j\right) - \left(x\frac{d}{dx} - j\right)\frac{d}{dx}\right) f \\
&= \left(\left(\frac{d}{dx} + x\frac{d^2}{dx^2} - j\frac{d}{dx}\right) - \left(x\frac{d^2}{dx^2} - j\frac{d}{dx}\right)\right) f \\
&= \frac{d}{dx} f,
\end{aligned}$$

$$\begin{aligned}
\left[\left(x\frac{d}{dx} - j\right), \left(2jx - x^2\frac{d}{dx}\right)\right] f &= \left(\left(x\frac{d}{dx} - j\right)\left(2jx - x^2\frac{d}{dx}\right)\right. \\
&\quad \left. - \left(2jx - x^2\frac{d}{dx}\right)\left(x\frac{d}{dx} - j\right)\right) f \\
&= \left(\left(2jx + 2jx^2\frac{d}{dx} - 2x^2\frac{d}{dx} - x^3\frac{d^2}{dx^2}\right)\right. \\
&\quad - \left(2j^2x - jx^2\frac{d}{dx}\right) - \left(2jx^2\frac{d}{dx} - 2j^2x\right) \\
&\quad \left. + \left(x^2\frac{d}{dx} + x^3\frac{d^2}{dx^2} - jx^2\frac{d}{dx}\right)\right) f \\
&= \left(2jx - x^2\frac{d}{dx}\right) f.
\end{aligned}$$

So we can build a relation between these two representations as:

$$2\left(x\frac{d}{dx} - j\right) \leftrightarrow M_d$$
$$\left(2jx - x^2\frac{d}{dx}\right) \leftrightarrow {M_1}^2$$
$$\frac{d}{dx} \leftrightarrow {M_2}^1$$

and they are different representations of the basis of the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.