Durham University
Mathematical Sciences Department
MSc Particles, Strings and Cosmology
Quantum Electrodynamics

Name: Rashad Hamidi Michaelmas Term 2021/2022

Homework

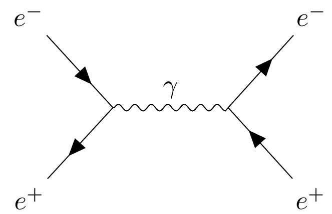
1 Bhabha Scattering

Draw all tree-level Feynman diagrams for Bhabha scattering ($e^+e^- \to e^+e^-$) and write down the amplitude for this process (do not compute the squared amplitude!)

Solution

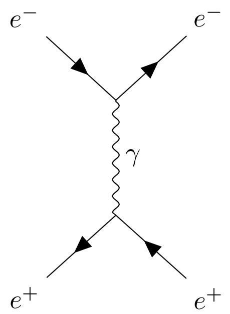
We need to find all tree-level Feynman diagrams for Bhabha scattering ($e^+(p)e^-(k) \to e^+(p')e^-(k')$). We have only two allowed diagrams at tree level,

s-channel



$$i\mathcal{M}_s = \bar{u}(k')(-ie\gamma^\mu)v(p')\frac{-ig_{\mu\nu}}{(k+p)^2}\bar{v}(p)(-ie\gamma^\nu)u(k).$$

t-channel



$$i\mathcal{M}_t = \bar{u}(p')(-ie\gamma^\mu)u(p)\frac{-ig_{\mu\nu}}{(k-k')^2}\bar{v}(k)(-ie\gamma^\nu)v(k').$$

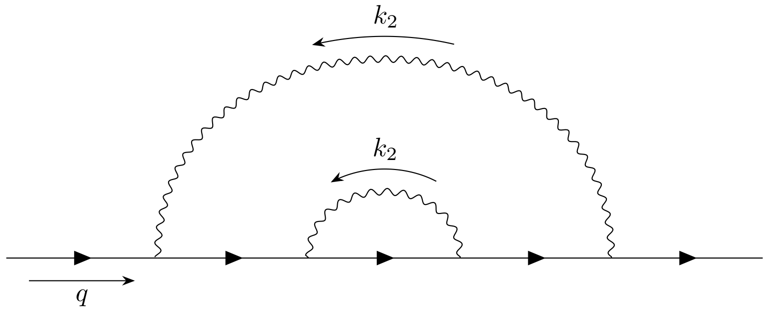
The total amplitude of this level is

$$\begin{aligned} i\mathcal{M}_2 &= i\mathcal{M}_s - i\mathcal{M}_t \\ &= ie^2\left[[\bar{u}(k')\gamma^\mu v(p')]\frac{1}{(k+p)^2}[\bar{v}(p)\gamma_\mu u(k)] - [\bar{u}(p')\gamma^\mu u(p)]\frac{1}{(k-k')^2}[\bar{v}(k)\gamma_\mu v(k')]\right], \end{aligned}$$

with a relative minus sign between the two diagrams.

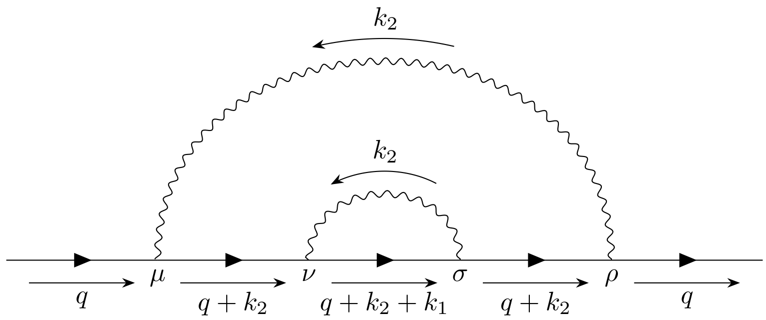
2 Loop Graphs

Write the amplitude for the following Feynman diagram without external wave function factors. Use the high energy limit.



Solution

Draw the momentum flow on the diagram,



Now, $i\mathcal{M}_{\text{Loops}}$ for a particle with a charge number Q, without the external wave function factors, at high energy limit ($m \sim 0$), and in D dimensions, is given by

$$
\begin{aligned}
i\mathcal{M}_{\text{Loops}} = \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_1}{(2\pi)^D} (-ieQ\gamma^\mu) \frac{i(\not{q} + \not{k}_2)}{(q+k_2)^2} (-ieQ\gamma^\nu) \frac{i(\not{q} + \not{k}_2 + \not{k}_1)}{(q+k_2+k_1)^2} (-ieQ\gamma^\sigma) \\
\frac{i(\not{q} + \not{k}_2)}{(q+k_2)^2} (-ieQ\gamma^\rho) \frac{-ig_{\mu\rho}}{k_2^2} \frac{-ig_{\nu\sigma}}{k_1^2},
\end{aligned}
$$

or,

$$
i\mathcal{M}_{\text{Loops}} = i(eQ)^4 \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_1}{(2\pi)^D} \gamma^\mu \frac{(\not{q} + \not{k}_2)}{(q+k_2)^2} \gamma^\nu \frac{(\not{q} + \not{k}_2 + \not{k}_1)}{(q+k_2+k_1)^2} \gamma^\sigma \frac{(\not{q} + \not{k}_2)}{(q+k_2)^2} \gamma^\rho \frac{g_{\mu\rho}}{k_2^2} \frac{g_{\nu\sigma}}{k_1^2}.
$$

3 Scalar Integrals

Using Feynman parametrisation, show that

$$I_{n,m}(p) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2]^n[(k+p)^2]^m} = (-p^2)^{\frac{D}{2}-m-n} C_{n,m}.$$

Hint: You may need the following identities:

$$\begin{aligned}
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)^n} &= \frac{i(-1)^n}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} (m^2)^{\frac{D}{2}-n}, \\
\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \\
\frac{1}{A^n B^m} &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{(1-x)^{n-1}x^{m-1}}{[(1-x)A+xB]^{n+m}}.
\end{aligned}$$

Solution

The first step to solve this integral is transforming it using Feynman parametrisation. The generalised Schwinger identity is

$$\prod_{i=1}^{m} a_i^{-n_i} = \frac{\Gamma\left(\sum_{i=1}^{m} n_i\right)}{\prod_{i=1}^{m}\Gamma(n_i)} \int \prod_{i=1}^{m} dx_i x_i^{n_i-1} \delta\left(1-\sum_{i=1}^{m} x_i\right)\left(\sum_{i=1}^{m} x_i a_i\right)^{-\sum_{i=1}^{m} n_i}.$$

Now, we have only two parameters,

$$a_1 = k^2, \qquad a_2 = (k+p)^2, \qquad n_1 = n, \qquad n_2 = m.$$

Then

$$a_1^{-n} a_2^{-m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int dx_1 dx_2 x_1^{n-1} x_2^{m-1} \delta(1-x_1-x_2)(x_1 a_1 + x_2 a_2)^{-n-m}.$$

Apply the integration over delta function i.e. $x_1 = 1 - x_2$,[1], set $x_2 = x$, then

$$a_1^{-n} a_2^{-m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{(1-x)^{n-1}x^{m-1}}{((1-x)a_1 + x a_2)^{n+m}}.$$

Substitute the values of a_1 and a_2, then

$$[k^2]^{-n}[(k+p)^2]^{-m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{(1-x)^{n-1}x^{m-1}}{((1-x)k^2 + x(k+p)^2)^{n+m}}.$$

[1]You can substitute $x_2 = 1 - x_1$ instead, but this will be somehow inconvenient in the following step, which is substituting the values of a_1 and a_2, because you will have one term with two times two squared terms needed to be expanded (i.e. 8 products).

Complete the square in the denominator,

$$[k^2]^{-n}[(k+p)^2]^{-m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{(1-x)^{n-1}x^{m-1}}{(k'^2-\Delta)^{n+m}},$$

where

$$k' = k + xp, \qquad \Delta = -(1-x)xp^2.$$

Back to our original integral, substitute the previous expression and flip the x integration with k' integration,[2]

$$I_{n,m}(p) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx (1-x)^{n-1}x^{m-1} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k'^2-\Delta)^{n+m}}.$$

We know the value of the integration over k' (from the hint), then

$$I_{n,m}(p) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \frac{i(-1)^{n+m}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n+m-\frac{D}{2})}{\Gamma(n+m)} \int_0^1 dx (1-x)^{n-1}x^{m-1}(\Delta)^{\frac{D}{2}-n-m}.$$

Substituting the value of Δ, then

$$I_{n,m}(p) = \frac{i(-1)^{n+m}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n+m-\frac{D}{2})}{\Gamma(n)\Gamma(m)} (-p^2)^{\frac{D}{2}-m-n} \int_0^1 dx (1-x)^{\frac{D}{2}-m-1} x^{\frac{D}{2}-n-1}.$$

The integration over x is Euler integral of the first kind, and its value is given in the hint, then

$$I_{n,m}(p) = (-p^2)^{\frac{D}{2}-m-n} \frac{i(-1)^{n+m}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n+m-\frac{D}{2})}{\Gamma(n)\Gamma(m)} \frac{\Gamma(\frac{D}{2}-n)\Gamma(\frac{D}{2}-m)}{\Gamma(D-n-m)},$$

or,

$$I_{n,m}(p) = (-p^2)^{\frac{D}{2}-m-n} C_{n,m},$$

where

$$C_{n,m} = \frac{i(-1)^{n+m}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n+m-\frac{D}{2})}{\Gamma(n)\Gamma(m)} \frac{\Gamma(\frac{D}{2}-n)\Gamma(\frac{D}{2}-m)}{\Gamma(D-n-m)}.$$

[2] Don't forget that $d^Dk' = d^Dk$

4 Tensor Integrals

Show the following identity

$$I^{\mu}_{n,m}(q) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{k^{\mu}}{[k^2]^n[(k+q)^2]^m} = -\frac{1}{2} q^{\mu} \frac{D-2m}{D-m-n} (-q^2)^{\frac{D}{2}-m-n} C_{n,m}$$

using Passarino-Veltman reduction.
Hint: You may need the following identity

$$C_{n,m-1} - C_{n-1,m} + C_{n,m} = \frac{D-2m}{D-m-n} C_{n,m}.$$

Solution

Using Passarino-Veltman reduction, the only possible ansatz is that which consists the tensor structure q^{μ}, i.e.

$$I^{\mu}_{n,m}(q) = c q^{\mu}.$$

To determine the coefficient c we now contract this ansatz with q_{μ},

$$q_{\mu} I^{\mu}_{n,m}(q) = q^2 c = \int \frac{d^D k}{(2\pi)^D} \frac{q_{\mu} k^{\mu}}{[k^2]^n[(k+q)^2]^m}.$$

We can now solve this integral by writing the numerator $q_{\mu} k^{\mu}$ as a combination of the propagators,

$$q_{\mu} k^{\mu} = \frac{1}{2}[(k+q)^2 - k^2 - q^2],$$

and express $q_{\mu} I^{\mu}_{n,m}(q)$ through scalar integrals only,

$$q_{\mu} I^{\mu}_{n,m}(q) = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \left[\frac{1}{[k^2]^n[(k+q)^2]^{m-1}} - \frac{1}{[k^2]^{n-1}[(k+q)^2]^m} - q^2 \frac{1}{[k^2]^n[(k+q)^2]^m} \right].$$

Using the result from the previous question,

$$q_{\mu} I^{\mu}_{n,m}(q) = \frac{1}{2} \left[(-q^2)^{\frac{D}{2}-m-n+1} C_{n,m-1} - (-q^2)^{\frac{D}{2}-m-n+1} C_{n-1,m} - q^2 (-q^2)^{\frac{D}{2}-m-n} C_{n,m} \right],$$

or,

$$q_{\mu} I^{\mu}_{n,m}(q) = \frac{1}{2} (-q^2)^{\frac{D}{2}-m-n+1} \left[C_{n,m-1} - C_{n-1,m} + C_{n,m} \right].$$

Using the given hint,

$$q_{\mu} I^{\mu}_{n,m}(q) = \frac{1}{2} (-q^2)^{\frac{D}{2}-m-n+1} \frac{D-2m}{D-m-n} C_{n,m},$$

which means that

$$c = -\frac{1}{2} (-q^2)^{\frac{D}{2}-m-n} \frac{D-2m}{D-m-n} C_{n,m}.$$

Finally,

$$I^{\mu}_{n,m}(q) = -\frac{1}{2} q^{\mu} \frac{D-2m}{D-m-n} (-q^2)^{\frac{D}{2}-m-n} C_{n,m}.$$

5 Loop Integral

Use the result of 4 to perform the two loop integrations in the amplitude found in 2.
Hints:

- Start with the k_1 integral.
- Use the identity $\gamma^\mu\gamma^\nu\gamma_\mu = (2-D)\gamma^\nu$
- Use the identity $\not{q}\not{q} = q^2$
- Express your result in terms of the constants $C_{n,m}$.

Solution

Recall the integral which we want to calculate,

$$i\mathcal{M}_{\text{Loops}} = i(eQ)^4 \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_1}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma^\nu \frac{(\not{q}+\not{k}_2+\not{k}_1)}{(q+k_2+k_1)^2} \gamma^\sigma \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma^\rho \frac{g_{\mu\rho}}{k_2^2} \frac{g_{\nu\sigma}}{k_1^2}.$$

Apply the metric tensor on gamma matrices,

$$i\mathcal{M}_{\text{Loops}} = i(eQ)^4 \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_1}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma^\nu \frac{(\not{q}+\not{k}_2+\not{k}_1)}{(q+k_2+k_1)^2} \gamma_\nu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma_\mu \frac{1}{k_2^2} \frac{1}{k_1^2}.$$

Using the second hint, contract γ^ν,

$$i\mathcal{M}_{\text{Loops}} = i(eQ)^4(2-D) \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_1}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \frac{(\not{q}+\not{k}_2+\not{k}_1)}{(q+k_2+k_1)^2} \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma_\mu \frac{1}{k_2^2} \frac{1}{k_1^2}.$$

Rearrange the terms,

$$\begin{aligned} i\mathcal{M}_{\text{Loops}} =& i(eQ)^4(2-D) \int \frac{d^D k_2}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \left[(\not{q}+\not{k}_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2(q+k_2+k_1)^2} \right. \\ &\left. + \int \frac{d^D k_1}{(2\pi)^D} \frac{\not{k}_1}{k_1^2(q+k_2+k_1)^2} \right] \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma_\mu \frac{1}{k_2^2}, \end{aligned}$$

or,

$$\begin{aligned} i\mathcal{M}_{\text{Loops}} =& i(eQ)^4(2-D) \int \frac{d^D k_2}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \left[(\not{q}+\not{k}_2) I_{1,1}(q+k_2) \right. \\ &\left. + \not{I}_{1,1}(q+k_2) \right] \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma_\mu \frac{1}{k_2^2}. \end{aligned}$$

Substitute the values of the integrations

$$\begin{aligned} i\mathcal{M}_{\text{Loops}} =& i(eQ)^4(2-D) \int \frac{d^D k_2}{(2\pi)^D} \gamma^\mu \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \left[(\not{q}+\not{k}_2)(-(q+k_2)^2)^{\frac{D}{2}-2} C_{1,1} \right. \\ &\left. - \frac{1}{2}(\not{q}+\not{k}_2)(-(q+k_2)^2)^{\frac{D}{2}-2} C_{1,1} \right] \frac{(\not{q}+\not{k}_2)}{(q+k_2)^2} \gamma_\mu \frac{1}{k_2^2}, \end{aligned}$$

or summing the terms and multiply the fractions,

$$i\mathcal{M}_{\text{Loops}} = \frac{i}{2}(eQ)^4(2-D)C_{1,1}\int\frac{d^Dk_2}{(2\pi)^D}\gamma^\mu\frac{(\not{q}+\not{k}_2)^3}{k_2^2[(q+k_2)^2]^2}\gamma_\mu(-(q+k_2)^2)^{\frac{D}{2}-2}.$$

Using the third hint, (i.e. $(\not{q}+\not{k}_2)^3 = (\not{q}+\not{k}_2)(q+k_2)^2)$, and then using the second hint to get rid of the γ^μ,

$$i\mathcal{M}_{\text{Loops}} = (-1)^{\frac{D}{2}}\frac{i}{2}(eQ)^4(2-D)^2C_{1,1}\int\frac{d^Dk_2}{(2\pi)^D}\frac{(\not{q}+\not{k}_2)}{k_2^2[(q+k_2)^2]^{3-\frac{D}{2}}}.$$

Rearrange the terms,

$$\begin{aligned} i\mathcal{M}_{\text{Loops}} =& (-1)^{\frac{D}{2}}\frac{i}{2}(eQ)^4(2-D)^2C_{1,1} \\ & \left[\not{q}\int\frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_2^2[(q+k_2)^2]^{3-\frac{D}{2}}} + \int\frac{d^Dk_2}{(2\pi)^D}\frac{\not{k}_2}{k_2^2[(q+k_2)^2]^{3-\frac{D}{2}}}\right], \end{aligned}$$

or,

$$i\mathcal{M}_{\text{Loops}} = (-1)^{\frac{D}{2}}\frac{i}{2}(eQ)^4(2-D)^2C_{1,1}\left[\not{q}I_{1,3-\frac{D}{2}}(q) + \not{I}_{1,3-\frac{D}{2}}(q)\right].$$

Substitute the values of the integrations

$$i\mathcal{M}_{\text{Loops}} = (-1)^{\frac{D}{2}}\frac{i}{2}(eQ)^4(2-D)^2C_{1,1}\left[\not{q}(-q^2)^{D-4}C_{1,3-\frac{D}{2}} - \frac{1}{2}\not{q}\frac{2D-6}{\frac{3}{2}D-4}(-q^2)^{D-4}C_{1,3-\frac{D}{2}}\right],$$

or summing the terms,

$$i\mathcal{M}_{\text{Loops}} = (-1)^{\frac{D}{2}}\frac{i}{2}(eQ)^4(2-D)^2\left[1-\frac{2D-6}{3D-8}\right]C_{1,1}C_{1,3-\frac{D}{2}}(-q^2)^{D-4}\not{q}.$$

Finished