DURHAM UNIVERSITY
Mathematical Sciences Department
MSc Particles, Strings and Cosmology
Quantum Field Theory I
Homework

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Problem 1 (Perturbation Theory)**.** Consider the simple harmonic oscillator, i.e. the Lagrangian

$$L_0 = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2, \tag{1}$$

its generating functional is

$$Z_0[J] = \exp\left(\frac{1}{2}\int_{-\infty}^{\infty} J(t_1)G_0(t_1,t_2)J(t_2)dt_1dt_2\right), \tag{2}$$

where G_0 is the two-point Green function for the harmonic oscillator. You should take this as the starting point for the exercise. Do not use the explicit form of G_0, use only that $G_0(t_1,t_2)$ is a symmetric function under the exchange $t_1 \leftrightarrow t_2$. In all computations leave the integrals over G_0 un-evaluated, so you should express all answers in terms of products and/or integrals of G_0.

a) Compute the generating functional

$$Z[J] = \mathcal{N}\int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\left(\frac{m}{2}\dot{q}(t)^2 - \frac{m\omega^2}{2}q(t)^2 - \lambda\frac{q(t)^4}{4!}\right) + \int_{-\infty}^{\infty} dt J(t)q(t)\right), \tag{3}$$

in perturbation theory to order λ. Do not forget to fix $\mathcal{N}$ to order λ.

b) Use the result of $Z[J]$, not forgetting $\mathcal{N}$, to compute the two-point Green function $G(t_1,t_2)$ to order λ.

Solution 1.

a) We can rewrite the generating functional as

$$\begin{aligned} Z[J] = &\exp\left(-\frac{i}{\hbar}\frac{\lambda}{4!}\int_{-\infty}^{\infty} dt\left(\frac{\delta}{\delta J(t)}\right)^4\right)\mathcal{N}\int\mathcal{D}q \\ &\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\left(\frac{m}{2}\dot{q}(t)^2 - \frac{m\omega^2}{2}q(t)^2\right) + \int_{-\infty}^{\infty} dt J(t)q(t)\right), \end{aligned} \tag{4}$$

or,

$$Z[J] = \frac{\mathcal{N}}{\mathcal{N}_0} \exp\left(-\frac{i}{\hbar}\frac{\lambda}{4!} \int_{-\infty}^{\infty} dt \left(\frac{\delta}{\delta J(t)} \right)^4 \right) Z_0[J]. \tag{5}$$

Expanding the functional exponential up to an order λ,

$$Z[J] = \frac{\mathcal{N}}{\mathcal{N}_0} \left(1 - \frac{i}{\hbar}\frac{\lambda}{4!} \int_{-\infty}^{\infty} dt \left(\frac{\delta}{\delta J(t)} \right)^4 + \mathcal{O}(\lambda^2) \right) Z_0[J]. \tag{6}$$

Now, evaluate the functional derivatives to obtain the generating functional to order λ,

$$\begin{aligned}
&\left(\frac{\delta}{\delta J(t)} \right)^4 \left[\exp\left(\frac{1}{2} J \cdot G_0 \cdot J \right) \right] \\
&= \left(\frac{\delta}{\delta J(t)} \right)^3 \left[(G_0 \cdot J)_t \exp\left(\frac{1}{2} J \cdot G_0 \cdot J \right) \right] \\
&= \left(\frac{\delta}{\delta J(t)} \right)^2 \left[\left(G_{0tt} + (G_0 \cdot J)_t^2 \right) \exp\left(\frac{1}{2} J \cdot G_0 \cdot J \right) \right] \\
&= \left(\frac{\delta}{\delta J(t)} \right) \left[\left(3G_{0tt} \left(G_0 \cdot J \right)_t + (G_0 \cdot J)_t^3 \right) \exp\left(\frac{1}{2} J \cdot G_0 \cdot J \right) \right] \\
&= \left(3G_{0tt}^2 + 6G_{0tt} \left(G_0 \cdot J \right)_t^2 + (G_0 \cdot J)_t^4 \right) \exp\left(\frac{1}{2} J \cdot G_0 \cdot J \right),
\end{aligned} \tag{7}$$

so we get,

$$Z[J] = \frac{\mathcal{N}}{\mathcal{N}_0} \left(1 - \frac{i}{\hbar}\frac{\lambda}{4!} \int dt \left(3G_{0tt}^2 + 6G_{0tt} \left(G_0 \cdot J \right)_t^2 + (G_0 \cdot J)_t^4 \right) + \mathcal{O}(\lambda^2) \right) Z_0[J]. \tag{8}$$

We should now fix the normalization by demanding that $Z[0] = \langle 0|0 \rangle = 1$, where as usual we assume the ground state is normalisable. We get,

$$Z[0] = \frac{\mathcal{N}}{\mathcal{N}_0} \left(1 - \frac{i}{\hbar}\frac{\lambda}{4!} \int dt 3G_{0tt}^2 + \mathcal{O}(\lambda^2) \right) = 1, \tag{9}$$

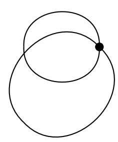
or,

$$\frac{\mathcal{N}}{\mathcal{N}_0} = \frac{1}{\left(1 - \frac{i}{\hbar}\frac{\lambda}{4!} \int dt 3G_{0tt}^2 + \mathcal{O}(\lambda^2) \right)}. \tag{10}$$

However, if we want the amplitude $\mathcal{N}/\mathcal{N}_0$ up to order λ, we can use the definition of the geometric series,

$$\frac{\mathcal{N}}{\mathcal{N}_0} = 1 + \frac{i}{\hbar}\frac{\lambda}{4!} \int dt 3G_{0tt}^2 + \mathcal{O}(\lambda^2), \tag{11}$$

absorbing the higher terms of λ and the signs in $\mathcal{O}(\lambda^2)$. Other representation of the amplitude factor $\mathcal{N}/\mathcal{N}_0$ using Feynman diagrams is

$\frac{\mathcal{N}}{\mathcal{N}_0} = 1 -$  $+ \mathcal{O}(\lambda^2),$ (12)

where we can see a vacuum bubble in the second term. This vacuum bubble will cancel another bubble with the same structure in the generating function. Plugging the value of the fixed amplitude into the generating function, and cancelling the vacuum bubble, then we have

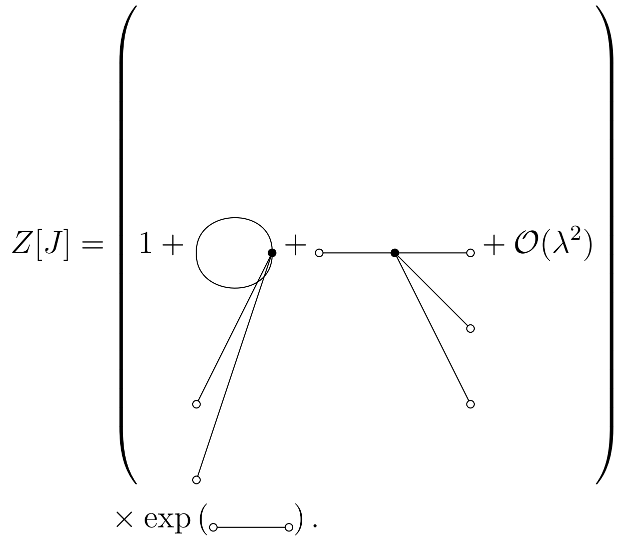
$$Z[J] = \left(1 - \frac{i}{\hbar}\frac{\lambda}{4!}\int dt\, \left(6G_{0tt}\,(G_0 \cdot J)_t^2 + (G_0 \cdot J)_t^4\right) + \mathcal{O}(\lambda^2)\right) Z_0[J], \qquad (13)$$

up to order λ.

b) *Now, we want to compute the two-point Green function $G(t_1, t_2)$ to order λ, which is basically*

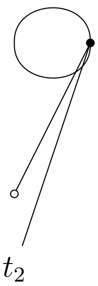
$$G(t_1, t_2) = \left.\frac{\delta^2 Z[J]}{\delta J(t_1)\delta J(t_2)}\right|_{J=0}. \qquad (14)$$

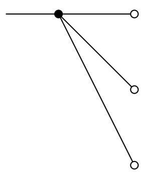
One can write the generating function using Feynman diagrams as:

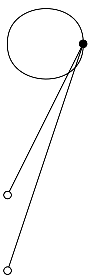
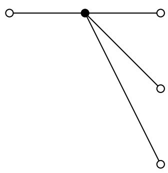


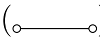
(15)

Then the two-point Green function $G(t_1, t_2)$ to order λ is given by

$G(t_1, t_2) \approx \frac{\delta}{\delta J_1} \Bigg($ 

$+\; t_2$ 

$+$ $t_2 \Bigg[1 +$  $+$  $\Bigg] \Bigg)$

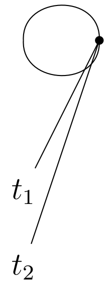
$\times \exp\big($  $\big)\Big|_{J=0},$ (16)

where the approximation sign because I removed the higher terms of λ, we can just add it in the final answer anyway. Now the final functional derivative, we can save time and typing, just by choosing the terms which contains one additional dot and plug t_1 into it, because

- *if the term do not have any dot then its functional derivative with respect to J is zero,*
- *if the term contains more than one dot then after evaluating at $J = 0$, the term automatically will be zero,*
- *and we do not need to derive the exponential function because it will give us an empty dot after the derivation and the evaluation at $J = 0$ will also automatically equal the term by zero.*

Hence, the only terms satisfy this condition are

$$G(t_1, t_2) = \; t_1 \text{ — } t_2 \; + \tag{17}$$



$$+\, \mathcal{O}(\lambda^2),$$

where the second term has a symmetry factor $1/2$ *and the first term does not have a symmetry factor. One can rewrite these diagrams of the two-points generating function as a mathematical representation by*

$$\begin{aligned} G(t_1, t_2) =& G_{0t_1t_2} - \frac{i}{\hbar}\frac{\lambda}{2}\int dt G_{0tt}G_{0tt_1}G_{0tt_2} + \mathcal{O}(\lambda^2) \\ \equiv& G_0(t_1, t_2) - \frac{i}{\hbar}\frac{\lambda}{2}\int dt G_0(t,t)G_0(t,t_1)G_0(t,t_2) + \mathcal{O}(\lambda^2), \end{aligned} \tag{18}$$

up to order λ*.*