Durham University
Mathematical Sciences Department
MSc Particles, Strings and Cosmology
Quantum Field Theory II

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Homework

1 ϕ^4 Theory

Write down the momentum space Feynman rules for the theory with action:

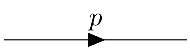
$$S[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right), \tag{1}$$

where ϕ is a real scalar field. Use these rules to write down the connected part of the scattering amplitude $i\mathcal{M}$ to order λ for two incoming particles with momenta p_1^μ and p_2^μ and two outgoing particles with momenta $p_{1'}^\mu$ and $p_{2'}^\mu$.

Solution

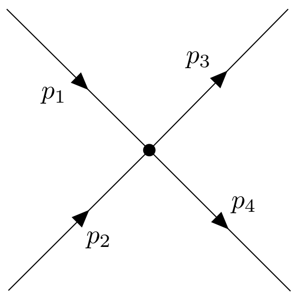
Feynman Rules

1. Propagator (Internal Line):



$$= \frac{i}{p^2 - m^2 + i\epsilon}. \tag{2}$$

2. Interaction Vertex:

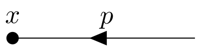


$$= -i\lambda. \tag{3}$$

3. Insertion (External Line/Leg):

Does not contribute.

4. External Point (External Vertex):



$$= e^{-ipx}. \tag{4}$$

5. Energy-Momentum Conservation at each Vertex:

$$(2\pi)^4\delta^{(4)}(\Sigma p_i - \Sigma p_f). \tag{5}$$

6. Integration over all Momenta:

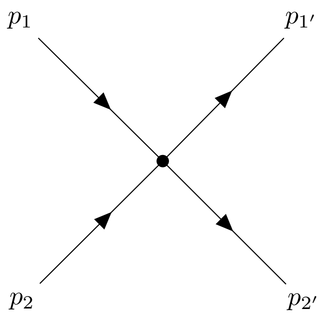
$$\int \frac{d^4p}{(2\pi)^4}. \tag{6}$$

7. Symmetry Factor:

Divide by a symmetry factor.

Contributed Connected Feynman Diagrams up to Order $\boldsymbol{\lambda}$

Using Feynman rules,



$$= -i\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_{1'} - p_{2'}), \tag{7}$$

and hence,

$$i\mathcal{M} \approx -i\lambda. \tag{8}$$

2 $0+0$ dimensional quantum field theory

Consider the one-dimensional integral

$$Z[\lambda] = \int dx \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4\right) \equiv Z[0]\exp(W[\lambda]), \tag{9}$$

which can be thought of as the partition function of an interacting quantum field theory in $0+0$ dimensions (i.e. with neither space nor time - there is only a single degree of freedom x).

(a) Consider performing a perturbative expansion of this partition function in powers of λ. This expansion can be organized in terms of Feynman diagrams. Write down the Feynman rules for this theory (i.e. what are the propagator and the interaction vertex?)

(b) Draw the Feynman diagrams contributing up to order λ^2 to $W[\lambda]$, which you recall from lecture to be the sum of connected diagrams. Being careful about factors (unlike in normal QFT, there is no other information!) evaluate the diagrams using the Feynman rules found previously.

(c) Use this to write down an expression for the ratio $Z[\lambda]/Z[0]$ up to order λ^2. Evaluate this approximate expression at $\lambda = 0.1$ and compare it to the exact ratio, which I have computed numerically to be 0.988306 or so.

Solution

(a) Firstly, we need to be careful for the meaning of each parameter. We do not have any space or time variables. Now, x is just a parameter and λ is our perturbation factor. Our action for this theory is given by

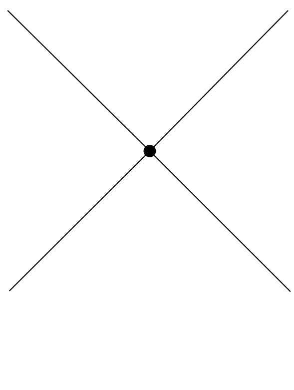
$$S[x;\lambda] = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4, \tag{10}$$

and the generating function is just

$$Z[\lambda] = \int dx \exp\left(-S[x;\lambda]\right). \tag{11}$$

Now, we do not have an x-semi-momentum term for this theory and the mass is $m \sim 1$. Hence, the propagator and the vertex in this theory are given by

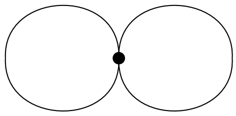
Propagator : $P =$ $= 1,$ (12)

Vertex : $V =$  $= -\lambda.$ (13)

(b) Now, Let's find our diagrams (vacuum bubbles):

$\mathcal{O}(\lambda)$

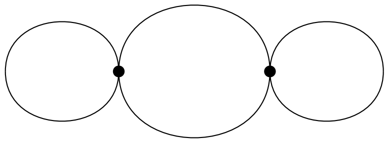
The first order ($\mathcal{O}(\lambda)$) has one diagrams



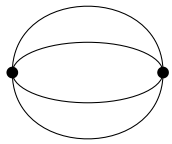
$$= \underbrace{\frac{1}{8}}_{\text{symm. fac.}} \underbrace{(-\lambda)}_{V} \underbrace{(1 \cdot 1)}_{P^2} = -\frac{\lambda}{8}. \tag{14}$$

$\mathcal{O}(\lambda^2)$

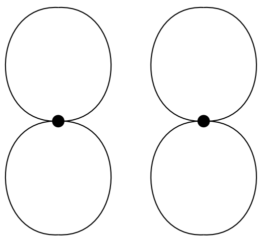
The second order ($\mathcal{O}(\lambda^2)$) has three diagrams



$$= \frac{1}{2^4} V^2 P^4 = \frac{\lambda^2}{16}, \tag{15}$$

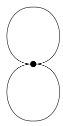
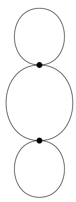
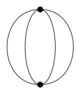
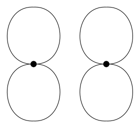


$$= \frac{1}{4! \cdot 2} V^2 P^4 = \frac{\lambda^2}{48}, \tag{16}$$



$$= \frac{1}{(2^3)^2 \cdot 2} V^2 P^4 = \frac{\lambda^2}{128}. \tag{17}$$

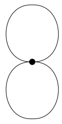
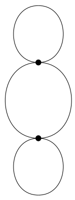
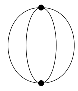
Hence, our generating function can be expanded as

$$Z[\lambda] = Z[0] \left(1 + \right.$$  $+$  $+$  $+$  $$\left. + \mathcal{O}(\lambda^3) \right). \tag{18}$$

Comment: Note that we have to take into account the connected and disconnected diagrams, because they both contribute in the generating function $Z[\lambda]$. The sum over only the connected diagrams for low orders in λ is asymptotic, not exact[1]. However, in the next part, we will see that calculating $Z[\lambda]$ as a sum of connected diagrams,

[1] I mean by exact, the exact contribution of that order without any cancellation due to the infinite expansion.

i.e.

$$Z[\lambda] = Z[0] \Bigg(1 +$$  $$+$$  $$+$$  $$+ \mathcal{O}(\lambda^3) \Bigg) , \tag{19}$$

is more precise than the summing of all connected and disconnected diagrams.

(c) The ratio $Z[\lambda]/Z[0]$ is given by

$$Z[\lambda]/Z[0] = \left(1 - \frac{\lambda}{8} + \left(\frac{1}{16} + \frac{1}{48} + \frac{1}{128}\right)\lambda^2 + \mathcal{O}(\lambda^3)\right), \tag{20}$$

and hence,

$$Z[0.1]/Z[0] \approx \left(1 - \frac{0.1}{8} + \left(\frac{1}{16} + \frac{1}{48} + \frac{1}{128}\right)(0.1)^2\right) = 0.988411458\bar{3}. \tag{21}$$

The exact value can be obtained be evaluating the integral

$$Z[\lambda]/Z[0] = \frac{\displaystyle\int dx \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4\right)}{\displaystyle\int dx \exp\left(-\frac{1}{2}x^2\right)} = \frac{1}{\sqrt{2\pi}} \int dx \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4\right). \tag{22}$$

However, there is an easier way to do that. The expression of $Z[\lambda]$ need to be modified, i.e. we have to add a new parameter, say j, s.t. if we take the functional derivatives of Z with respect to j in the form of the perturbative part, then it have to give us $Z[\lambda]$ at $j = 0$. Hence, our new generating function is given by

$$Z[\lambda; j] = \int dx \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + jx\right), \tag{23}$$

s.t.

$$Z[\lambda] = Z[\lambda; 0]. \tag{24}$$

We can now write the generating functional as

$$Z[\lambda; j] = \exp\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4\right) \int dx \exp\left(-\frac{1}{2}x^2 + jx\right). \tag{25}$$

Completing the square of the exponentiated part give us

$$Z[\lambda; j] = \exp\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4\right) \exp\left(\frac{j^2}{2}\right) \int dx \exp\left(-\frac{1}{2}(x - j)^2\right). \tag{26}$$

Integrating over x of the Gaussian function leads to

$$Z[\lambda; j] = \sqrt{2\pi} \exp\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4\right) \exp\left(\frac{j^2}{2}\right), \tag{27}$$

or

$$Z[\lambda; j] = Z[0] \exp\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4\right) \exp\left(\frac{j^2}{2}\right), \tag{28}$$

where (by evaluating the Gaussian integral)

$$Z[0] = \sqrt{2\pi}. \tag{29}$$

Now, using Equation (28),

$$Z[\lambda]/Z[0] = Z[\lambda; j]/Z[0]|_{j=0} = \exp\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4\right) \exp\left(\frac{j^2}{2}\right)\Bigg|_{j=0}. \tag{30}$$

Expand the exponentiated functional derivative and the exponential in j,

$$Z[\lambda]/Z[0] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)}\left(-\frac{\lambda}{4!}\right)^n \left(\frac{\delta}{\delta j}\right)^{4n} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left(\frac{1}{2}\right)^k j^{2k}\Bigg|_{j=0}. \tag{31}$$

Apply the functional derivative,

$$Z[\lambda]/Z[0] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)}\left(-\frac{\lambda}{4!}\right)^n \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left(\frac{1}{2}\right)^k \frac{\Gamma(2k+1)}{\Gamma(2k-4n+1)} j^{2k-4n}\Bigg|_{j=0}. \tag{32}$$

Evaluate at $j = 0$, then the only survival term of the summation over k is when $k = 2n$,

$$Z[\lambda]/Z[0] = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(4n+1)}{(4!)^n 2^{2n} \Gamma(n+1)\Gamma(2n+1)} \lambda^n. \tag{33}$$

If you replace the infinity over the summation by N, then you can calculate the ratio $Z[\lambda]/Z[0]$ approximately up to order N,

$$Z[\lambda]/Z[0]|_{\mathcal{O}(\lambda^N)} = \sum_{n=0}^{N} \frac{(-1)^n \Gamma(4n+1)}{(4!)^n 2^{2n} \Gamma(n+1)\Gamma(2n+1)} \lambda^n. \tag{34}$$

If you substitute $N = 2$, then you will get the same result as I computed previously. However, the sum over only the connected diagrams is

$$Z[0.1]/Z[0]|^{\text{asymptotic}}_{\mathcal{O}(\lambda^2)} = \left(1 - \frac{0.1}{8} + \left(\frac{1}{16} + \frac{1}{48}\right)(0.1)^2\right) = 0.988\bar{3}, \tag{35}$$

for counting both connected and disconnected diagrams up to order λ^2 is

$$Z[\lambda]/Z[0]|^{\text{exact}}_{\mathcal{O}(\lambda^2)} = \sum_{n=0}^{2} \frac{(-1)^n \Gamma(4n+1)}{(4!)^n 2^{2n} \Gamma(n+1)\Gamma(2n+1)} \lambda^n = 0.988411458\bar{3}, \tag{36}$$

and the exact value for all terms is

$$Z[0.1]/Z[0]|_{\mathcal{O}(\lambda^\infty)} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n \Gamma(4n+1)}{(4!)^n 2^{2n} \Gamma(n+1)\Gamma(2n+1)} \lambda^n = 0.988306. \tag{37}$$

Hence,

$$Z[\lambda]/Z[0]|^{\text{exact}}_{\mathcal{O}(\lambda^2)} \gtrsim Z[0.1]/Z[0]|^{\text{asymptotic}}_{\mathcal{O}(\lambda^2)} \gtrsim Z[0.1]/Z[0]|_{\mathcal{O}(\lambda^\infty)}. \tag{38}$$

Comment: It is interesting that the asymptotic approximation (the sum over the connected diagrams only) is closer than the exact value (the sum over the connected and disconnected diagrams) of that order contribution to the exact value of the functional $W[\lambda]$ to order infinity. Is it weird that the asymptotic value being better that the exact value? No; because the meaning of asymptotic and exact is for the order by order contribution, not which of them is closer to the exact value at infinity (which is the same as the asymptotic value at infinity).

3 $SO(N)$ Symmetry

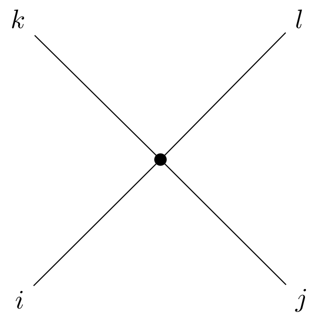
In this problem we will study a theory of N self-interacting scalar fields ϕ^i, $i \in \{1, \ldots, N\}$ with an $SO(N)$ symmetry. Consider the action

$$S[\phi^i] = \int d^4x \left(\frac{1}{2}(\partial\phi^i)^2 - V(\phi^i) \right), \tag{39}$$

where $(\phi^i)^2 = \phi^i\phi^i$ (i.e. the magnitude-squared of the N-component vector ϕ^i), and where

$$V(\phi^i) = \frac{m^2}{2}(\phi^i)^2 + \frac{\lambda}{4}((\phi^i)^2)^2. \tag{40}$$

(a) First, consider the theory with $m^2 > 0$. Write down the Feynman rules for this theory. In particular, show that there is one type of vertex, and that it is as shown in Figure 1 (i.e. the vertex is $-2i\lambda$ between e.g. two ϕ^1's and two ϕ^2's, but is $-6i\lambda$ between four ϕ^1's.)



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl})$$

Figure 1. Vertex in $SO(N)$-invariant theory when $m^2 > 0$.

(b) Now consider the theory with $m^2 < 0$, i.e. we write $m^2 = -\mu^2$, where $\mu^2 > 0$. Now the point in the potential V with $\phi^i = 0$ is a local maximum rather than a minimum, and thus the equilibrium value of ϕ^i will now have a nonzero value that minimizes the potential. By an $SO(N)$ rotation we can choose this nonzero value to point in the $i = N$ direction: thus we single out the $i = N$ component of ϕ^i and write the theory in terms of new fields σ and π^i:

$$\phi^N(x) = v + \sigma(x), \tag{41}$$
$$\phi^i(x) = \pi^i(x), \qquad i \in \{1, \ldots, N-1\}, \tag{42}$$

where v is a constant that is picked to minimize V. What is the value of v? What are the masses of the π^i and σ fields?

(c) Work out the self interactions of the σ and π^i fields and write down the Feynman rules (i.e. the new propagators and vertices) in terms of these fields. Calculate (to order λ) the connected part of the matrix element $i\mathcal{M}$ for the scattering of $\pi^1\pi^1 \to \pi^2\pi^2$, where the incoming particles have momenta p_1, k_1 and the outgoing particles have momenta p_2, k_2. Express all your answers in terms of the parameters in the Lagrangian μ and λ.

Solution

(a) The interaction term is given by

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}((\phi^i)^2)^2. \tag{43}$$

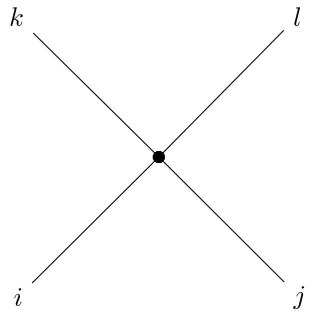
This term can be expanded as

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\delta^{(ij}\delta^{kl)}\phi^i\phi^j\phi^k\phi^l, \tag{44}$$

or expanding over the total symmetric tensor,

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\frac{1}{3}(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl})\phi^i\phi^j\phi^k\phi^l, \tag{45}$$

where δ^{uv} is just a Kronecker delta[2]. The second thing we have to take into account is the symmetry multiplier. According to combinatorics science, the vertex has four identical legs, and consequently it has 4! way of rearranging the legs, which has to be multiplied to the vertex value. Hence, (don't forget the imaginary number i outside the integral in the generating function) the vertex is given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}). \tag{46}$$

The fields in this theory have a kinetic and mass terms, hence the propagator is given by

$$i \;\text{———}\; j \; = \frac{i\delta^{ij}}{p^2 - m^2 + i\epsilon}. \tag{47}$$

The other Feynman rules are the same as the ones mentioned in the first problem. The vertex connect two similar fields to other two similar fields, we cannot have other than this combination of fields. Thus, if we have two similar fields, then the vertex amplitude is $-2i\lambda$, while if we have four similar vertices, then the vertex amplitude is $-6i\lambda$.

(b) To explain what is happening here, let's do some graphing and plot the potential $V(\phi^i)$ vs the norm of the vector field $||\phi^i||$. It is easy to draw this graph by hand, but however, I will use Wolfram Mathematica to do so. Write the following code on Wolfram Mathematica kernel,

[2] If ϕ^i was a Lorentz vector, then we have to use the metric tensor $-g^{\mu\nu}$ instead of Kronecher delta δ^{uv}, taking into account the upper and lower indices.

```
In[•]:= Plot[{4 x^2 + x^4, -4 x^2 + x^4}, {x, -5, 5},
         AxesLabel → {Norm[φ^i], V[φ^i]}, PlotRange → 5, PlotLegends → {m^2, μ^2}]
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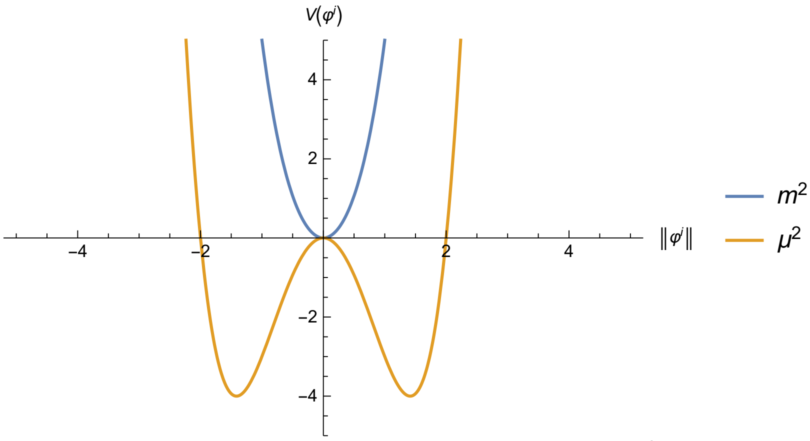
Out[•]=



As we see, the plot[3] in the case of positive mass term $+m^2$, we have on local minimum without any local maxima. On the other hand, if we have a negative mass term $-\mu^2$, then we will have a nonzero "surface" local minimum[4] while at zero there will be a "point" local maximum. Now, by an $SO(N)$ rotation we can choose this nonzero value (the value of $||\phi^i||$) to point in the $i = N$ (the last component) direction: thus we single out the $i = N$ component of ϕ^i and write the theory in terms of new fields σ and π^i as given in the problem. Then v must be picked to minimise V. To do that, we have to find the surface local minimum, using the usual technique of minimisation and maximisation[5], take the derivative of the potential with respect to $(\phi^i)^2$ and equal the answer to zero, i.e.

$$\left(\frac{\partial V(\phi^i)}{\partial(\phi^j)^2}\right)_0 = -\frac{\mu^2}{2}\delta^{ij} + \frac{\lambda}{2}\left(\phi_0^i\right)^2\delta^{ij} = 0, \tag{48}$$

and hence, the surface local minimum at

$$(\phi_0^j)^2 = \frac{\mu^2}{\lambda}. \tag{49}$$

Thus, this value is picked to be our v^2,

$$v = \frac{\mu}{\sqrt{\lambda}}. \tag{50}$$

If we replaced the mass $+m^2$ by the mass $-\mu^2$, then Lagrangian in this case is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi^i)^2 + \frac{\mu^2}{2}(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2. \tag{51}$$

[3] Don't care about the fact that $||\phi^i||$ must be positive because all we need from this plot is to show that the minimum is not at zero, while we have a maximum there.

[4] A "surface" local minimum means, if $\dim(\phi^i) = D$, then the "surface" local minimum will be in $D-1$ dimensional space, specifically S^{D-1}, $(D-1)$-sphere, this is because of the symmetricity of the potential.

[5] There is another method to find the minimum value, which can be accomplished by completing the square of the potential function. The constant value which minimise the potential is given to be v^2 in this theory (i.e. we will have the form $\lambda((\phi^i)^2 - v^2)^2/4 -$ constant, then v^2 is picked to equal $(\phi^i)_0^2$ at the minimum value).

Plug our new fields in the Lagrangian, then

$$\mathcal{L} = \frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{\mu^2}{2}((\pi^i)^2 + (v+\sigma)^2) - \frac{\lambda}{4}((\pi^i)^2 + (v+\sigma)^2)^2. \tag{52}$$

Expand everything in the Lagrangian,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{\mu^2}{2}(\pi^i)^2 + \frac{\mu^2}{2}\sigma^2 + \mu^2 v\sigma + \frac{\mu^2}{2}v^2 \\ &- \frac{\lambda}{4}(((\pi^i)^2)^2 + 2\sigma^2(\pi^i)^2 + 4v\sigma(\pi^i)^2 + 2v^2(\pi^i)^2 \\ &+ (\sigma^2)^2 + 4v\sigma^3 + 6v^2\sigma^2 + 4v^3\sigma + v^4).\end{aligned} \tag{53}$$

Combine the similar terms as next

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{\mu^2}{2}(\pi^i)^2 + \frac{\mu^2}{2}\sigma^2 + \mu^2 v\sigma + \frac{\mu^2}{2}v^2 \\ &- \frac{\lambda}{4}(((\pi^i)^2)^2 + 2\sigma^2(\pi^i)^2 + 4v\sigma(\pi^i)^2 + 2v^2(\pi^i)^2 \\ &+ (\sigma^2)^2 + 4v\sigma^3 + 6v^2\sigma^2 + 4v^3\sigma + v^4),\end{aligned} \tag{54}$$

other terms are singlets. Substitute the value of v and combine the terms,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2 - \mu^2\sigma^2 + \frac{\mu^4}{4} \\ &- \frac{\lambda}{4}(((\pi^i)^2)^2 + 2\sigma^2(\pi^i)^2 + 4v\sigma(\pi^i)^2 + (\sigma^2)^2 + 4v\sigma^3).\end{aligned} \tag{55}$$

From this Lagrangian, one can deduce that the field σ is massive, while the fields π^i are massless

$$m_\sigma = \sqrt{2}\mu, \tag{56}$$

$$m_{\pi^i} = 0, \qquad i \in \{1, \dots, N-1\}, \tag{57}$$

and this is because there is no mass term in the form constant $\times$ mass $\times$ (field)^2 in the Lagrangian for the fields π^i, while there is one for the field σ.

(c) The Lagrangian parts are given by

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2 \tag{58}$$

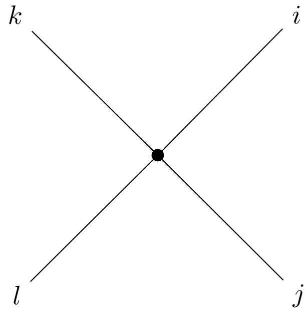
$$\mathcal{L}_{\text{mass}} = -\mu^2\sigma^2 \tag{59}$$

$$\mathcal{L}_{\text{const}} = \frac{\mu^4}{4} \tag{60}$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}(((\pi^i)^2)^2 + (\sigma^2)^2 + 2\sigma^2(\pi^i)^2 + 4v\sigma(\pi^i)^2 + 4v\sigma^3) \tag{61}$$

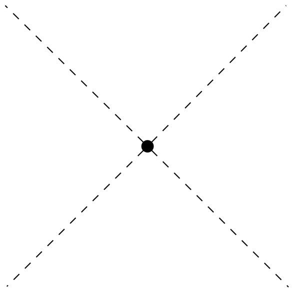
Now, the first two terms of the interaction term are total self-interaction terms and can be treated as perturbations for small λ. The third field is a mixed-interaction term and can be treated as perturbations for small λ. On the other hand, we cannot be sure that the last two terms are perturbations or not because they depend on the mass of the σ field, so if it is a super-massive field, the terms will no longer be perturbations. Hence, we have 5 terms in our interaction theory, and hence we have 5 types of vertices:

$((\pi^i)^2)^2$:



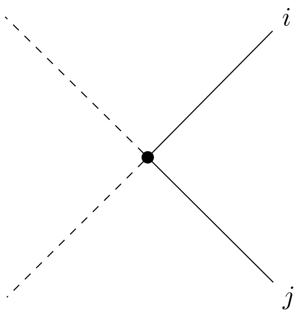
$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) \tag{62}$$

$(\sigma^2)^2$:



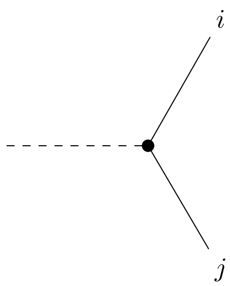
$$= -6i\lambda \tag{63}$$

$\sigma^2(\pi^i)^2$:



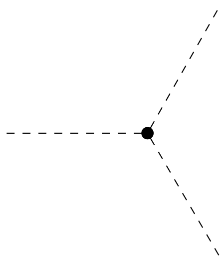
$$= -2i\lambda\delta^{ij} \tag{64}$$

$\sigma(\pi^i)^2$:



$$= -2i\mu\sqrt{\lambda}\delta^{ij} \tag{65}$$

σ^3 :



$$= -6i\mu\sqrt{\lambda} \tag{66}$$

We also have two different propagators for this theory,

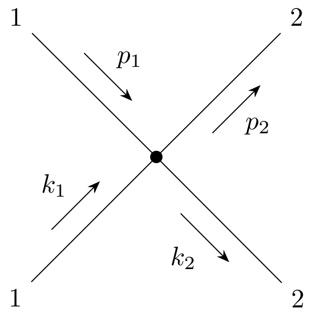
$$\pi^i \to \pi^j : \qquad i \longrightarrow j \;= \frac{i\delta^{ij}}{p^2 + i\epsilon} \tag{67}$$

$$\sigma \to \sigma : \qquad \text{- - - - - - - - - - - -} = \frac{i}{p^2 - 2\mu^2 + i\epsilon} \tag{68}$$

$\pi^1\pi^1 \to \pi^2\pi^2$ Scattering

Now, we will use the new Feynman rules to write the matrix element $i\mathcal{M}$ to the first order in λ for the scattering $\pi^1\pi^1 \to \pi^2\pi^2$, where the incoming particles have momenta p_1, k_1 and the outgoing particles have momenta p_2, k_2. In the first order in λ, we only have the four-legs vertices, and specifically the $((\pi^i)^2)^2$ vertex, all other interaction terms vanish in the first order perturbation. Hence,

$$i\mathcal{M}_{\pi^1\pi^1\to\pi^2\pi^2} =$$



$$= -2i\lambda. \tag{69}$$

Remember that we used

$$(\delta^{11}\delta^{22} + \delta^{12}\delta^{12} + \delta^{12}\delta^{12}) = 1. \tag{70}$$

Finished